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ABSTRACT

In previous papers the author and C.U. Martel have proposed a "poly-matroidal" network flow model and shown how it can be applied to obtain simple proofs of a number of duality theorems in polyhedral combinatorics. In the present paper the model is generalized to permit capacity constraints and lower bound constraints to be specified by intersecting families of sub-sets of arcs. It is shown how this generalization of the model can be used to obtain a particularly simple proof of the discrete separation theorem of Frank. A procedure is described for obtaining a feasible flow in a network with intersecting families of capacity constraints and lower bound constraints provided these constraints are in a relation of "compliance." It is shown how the Edmonds-Giles problem can be formulated as a minimum-cost circulation problem and how, with a single maximal flow computation, a feasible solution to the Edmonds-Giles problem can be determined.

1. INTRODUCTION

Network flow theory, as developed in the 1950's and early 1960's by FORD & FULKERSON [3] and others, has achieved "classical" status. There are many reasons for the enduring importance of this theory. The network flow model is simple, intuitively appealing, and highly versatile. The theory provides easily comprehended general theorems from which important results follow as simple corollaries. Proof techniques tend to be constructive, that is to say algorithmic, in keeping with today's orientation toward computation.

The past decade has seen a flowering of new results in polyhedral combinatorics. Though many of these results can be viewed as generalizations of theorems of network flow theory, there has been relatively little effort to interpret them in that light. The dominant paradigm has been strictly algebraic. That is, a linear programming problem is formulated, integrality properties of the problem and its dual are established and certain combinatorial duality relations, often graph-theoretic in character, are deduced as a result.

It is the conviction of the author that there is much value to be gained from generalizing the classical network flow model to accomodate the newer results of polyhedral combinatorics. Some preliminary work, as in LAWLER & MARTEL [9,10] has yielded encouraging results.

In this paper we begin by generalizing the source-sink model of [9,10] to permit capacity constraints to be specified by intersecting families of subsets of arcs. It is shown now the max-flow min-cut theorem of [9] generalizes to this case and now this yields a particularly simple proof of the "discrete separation theorem" of FRANK [4].

We then generalize the model to allow nonzero lower bounds on flow to be specified by functions which are supermodular on intersecting families of arcs. It is shown that capacity constraints and lower bound constraints cannot be in an arbitrary relation to each other, else an intractible problem results. The notion of "compliance" is introduced and the max-flow min-cut theorem is generalized to the case of compliant capacity and lower bound constraints. It is shown how feasible flows can be computed for networks with such constraints. The Edmonds-Giles problem is formulated as

a circulation problem and it is shown now a feasible vector in the Edmonds-Giles polyhedron can be obtained with a single maximal flow computation.

It should be noted that several proofs, particularly of variations of the max-flow min-cut theorem, are omitted from this paper. These will be provided in a later version intended for publication.

2. SOME PRELIMINARIES

We are concerned with a generalization of classical network flow theory in which constraints are applied to the total flow through certain specified subsets of arcs. These constraints must be in some sense submodular.

In this section we provide some needed background and terminology.

Let E be a finite set of arcs. (Ordinarily, E will be $A_i^+(A_i^-)$, the set of arcs directed into (out of) a given node i. Later, we shall let $E = A_i^+ \cup A_i^-$.) Let C be a collection of subsets of E and c be a real-valued set function assigning a *capacity* c(S) to each $S \in C$, with $c(\emptyset) = 0$ if $\emptyset \in C$. If, for given X and Y in C, it is the case that $X \cup Y$ and $X \cap Y$ are in C and

$$c(X) + c(Y) \geq c(X \cup Y) + c(X \cap Y),$$

then c is *submodular* on X and Y. If, for all X and Y in C, $X \subseteq Y$ implies $c(X) \leq c(Y)$ then c is *monotone*.

For each e in E, let x(e) be an amount of *flow*. A flow x is said to be *feasible* with respect to the capacity constraints (C, c) if $x(Y) \leq c(Y)$ for all Y in C.

Suppose $C = 2^E$, the power set of E, and c is an (integer) monotone function submodular on all pairs X and Y in C. Then the nonnegative flows feasible with respect to (C, c) are precisely the feasible vectors of the (integer) *polymatroid*

$$P = \{x \mid x \geq 0, \ x(Y) \leq c(Y) \ \text{for} \ Y \in 2^E\}.$$

Suppose c is nonnegative and submodular but not monotone. Then the function c', where

$$c'(X) = \min\{c(Y) \mid X \subseteq Y\}$$

is monotone and submodular and defines the same set of nonnegative flows as c. In any case, if c is submodular over all pairs, but not necessarily nonnegative, then the flows feasible with respect to (C,c) are the feasible vectors of the *extended polymatroid*

$$P' = \{x \mid x(Y) \leq c(Y) \quad \text{for} \quad Y \in 2^E\}.$$

Let us consider situations in which C may be a proper subset of 2^E and in which the function c is not necessarily submodular on all pairs of subsets in C. We consider three possibilities, in increasing order of generality.

C is a *ring family*: if X and Y are in C then X ∪ Y and X ∩ Y are in C and c is submodular on X and Y. If c is nonnegative, then the nonnegative flows feasible with respect to (C,c) are the feasible vectors of some polymatroid [1]. It is also known that all flows feasible with respect to (C,c) are the feasible vectors of some extended polymatroid.

C is an *intersecting family*: If X and Y are in C and X ∩ Y ≠ ∅, then X ∪ Y and X ∩ Y are in C and c is submodular on X and Y. (Such X and Y are said to be an *intersecting pair*.) For every intersecting family C, there is an equivalent ring family C', obtained as follows. Let C' be the family of subsets X ⊆ E which can be represented as the union of disjoint sets $X_1,...,X_t$ (for some t ≥ 0) in C, i.e.

$$C' = \{X \mid X = \cup X_i, \ X_i \cap X_j = \emptyset, \ X_i \in C\} \cup \{\emptyset\}.$$

For each X ∈ C' − {∅}, let

(2.1) $c'(X) = \min\{\Sigma c(X_i)\},$

where the minimum is taken over all disjoint X in C such that $X = \cup X_i$. Define c(∅) = 0. Then C' is a ring family and c' is submodular on all intersecting pairs. Moreover, the flows feasible with respect to (C',c') are precisely those which are feasible with respect to (C,c).

C is a *crossing family*: If X and Y are in C and $X \cap Y = \emptyset$, $X \cup Y \neq E$, then $X \cup Y$ and $X \cap Y$ are in C and c is submodular on X and Y. (Such X and Y are said to be a *crossing pair*.) If flows are subject to an additional constraint of the form $x(E) = k$, then it is possible to obtain an equivalent intersecting family C' as follows [4,11]: Let

$$C' = \{X \mid X = \cap X_i \neq \emptyset, \ \bar{X}_i \cap \bar{X}_j = \emptyset, \ X_i \in C\} \cup \{E\}.$$

For each $X \in C' - \{E\}$, let

$$c'(X) = k - \max\{\Sigma(K-c(X_i))\},$$

where the maximum is taken over all X_i such that $X = \cap X_i$ and $\bar{X}_i \cap \bar{X}_j = \emptyset$. Define $c'(k) = k$. Then C' is an intersecting family and c' is submodular on intersecting pairs. Moreover those flows which are feasible with respect to (C',c') and the additional constraint $x(E) = k$ are those which are feasible for (C,c) and the same additional constraint.

It is seen that for each of these types of families a sequence of transformations can be made to yield an equivalent extended, or possibly nonextended, polymatroid. (Note that each of these transformations preserves integrality of the capacity functions.) We could choose to describe our network flow model in terms of any of these families (provided there is an additional constraint in the case of a crossing family). In [9,10] constraints were assumed to be defined directly by polymatroids. In this paper, we shall describe the network flow model in terms of intersecting families of constraints.

Suppose x is a flow which is feasible with respect to a given intersecting family of constraints (C,c). A set Y in C is said to be *tight* (or "saturated") with respect to x if $x(Y) = c(Y)$. A crucial consequence of submodularity of c on intersecting sets is that the class of tight sets containing a given arc e is closed under union and intersection. It follows that if c is contained in any tight set, then there is a *unique minimal* tight set in C which contains e. Furthermore, if x is nonnegative and c is monotone then this unique minimal tight set contains no arc $e' \neq e$ which

is void, i.e. an arc e' such that x(e') = 0. (Cf. [9].)

3. A SIMPLE SOURCE-SINK MODEL

We begin with a simple source-sink model, similar to that in [9,10]. The network is defined over a directed multigraph (multiple arcs from one node to another are permitted) with a designated *source* s and *sink* t. For each node i let A_i^+ (A_i^-) denote the set of arcs directed into (out of) i. Two intersecting families of constraints (C_i^+, c_i^+), (C_i^-, c_i^-) are defined for each node i, where $C_i^+ \subseteq 2^{A_i^+}$, $C_i^- \subseteq 2^{A_i^-}$, and each of the functions c_i^+, c_i^- is nonnegative.

The problem is to find a flow x of maximum value. That is,

$$\text{maximize } v = x(A_s^-) - x(A_s^+) = x(A_t^+) - x(A_t^-)$$

subject to

$$x(A_i^+) - x(A_i^-) = 0 \qquad \text{for } i \neq s,t,$$

$$x(Y) \leq c_i^+(A_i^+) \qquad \text{for all i and } Y \in C_i^+$$

$$x(Y) \leq c_i^-(A_i^-) \qquad \text{for all i and } Y \in C_i^-$$

$$x \geq 0.$$

We now summarize the results of [9] which apply to this problem.

With respect to a given feasible flow x, i.e. a flow satisfying all the constraints of the maximum flow problem, an *augmenting path* is an undirected path of distinct arcs (but not necessarily distinct nodes) from s to t such that
(3.1) each backward arc e in the path is nonvoid, i.e. x(e) > 0, and
(3.2) if the head (tail) of a forward arc e in the path is contained in a tight set at its head (tail), then the following (preceding) arc in the path is a backward arc contained in its unique minimal tight set.

It is shown in [9] that for any augmenting path there exists a strictly positive value δ by which the existing feasible flow can be augmented.

THEOREM 3.1 (Augmenting path theorem)

A flow is maximal if and only if it admits of no augmenting path.

Augmenting paths can be found by a labeling procedure similar to that of ordinary network flow theory, except that arcs rather than nodes are labeled.

THEOREM 3.2 *A maximal flow can be achieved with at most m^3 augmentations, where m is the number of arcs in the network, provided each successive augmenting path contains as few arcs as possible, with ties between shortest paths being broken by a lexicographic ordering rule.*

The preceding theorem, which can be compared to a similar result for the ordinary network flow model, holds without regard to integrality of the capacity functions. If the capacity functions are integral, we have the following.

THEOREM 3.3 (Integrality theorem)

If all capacity functions are integral valued, then there is a maximal flow which is integral.

The significant modification of the results in [9] which is required for the generalization from polymatroidal constraints to intersecting constraints is in the statement of the max-flow min-cut theorem. The dual structure we shall consider is of the form (S,T,L,U), where S and T, with $s \in S$, $t \in T$, define a bipartition of the nodes of the network defining a cut and L and U define a bipartition of (all) the arcs of the network. (Note: L,U are *not* necessarily the subsets of labeled and unlabeled arcs at the end of the maximum flow computation, but can be derived from these subsets.) We define the *capacity* of any such (S,T,L,U) as

$$c(S,T,L,U) = \sum_{i \in S} d_i^-(A_i^- \cap U) + \sum_{j \in T} d_j^+(A_j^+ \cap L),$$

where

$$(3.3) \qquad d_i^-(X) = \min\{\sum c_i^-(X_k)\},$$

(3.4) $d_j^+(X) = \min\{\sum c_j^+(X_k)\}$,

where, as in (2.1), the minimizations are taken over all disjoint X_k in c_i^+ or C_j^- such that $X = \cup X_k$.

It should be clear that for any feasible flow x and any dual structure (S,T,L,U), the value v of x is such that $v \leq c(S,T,L,U)$. The max-flow min-cut theorem below asserts that equality can be obtained.

THEOREM 3.4 (Max-flow min-cut theorem)

The maximum value of a flow is equal to the minimum capacity of a dual structure. Moreover,

(i) *if all capacity constraints are specified by ring families, then*
     $d_i^+ = c_i^+$, $d_j^- = c_j^-$ *in* (3.3), (3.4)

(ii) *if all capacity constraints are specified by polymatroids, then the dual structures may be taken to be arc-partitioned cuts, i.e. L and U define a bipartition of the forward arcs across the cut* (S,T).

4. AN APPLICATION OF THE SOURCE-SINK MODEL

In [10], it is shown that a variety of known duality results, including Edmonds' polymatroid duality theorem and the Edmonds-Fulkerson theorem of matroid partitioning, are almost immediate corollaries of the integrality theorem and the max-flow min-cut theorem. These derivations exactly parallel the way in which, for example, Menger's theorem and the Konig-Egervary theorem are obtained in ordinary network flow theory. Rather than repeat these results here, we propose to show how one can obtain a particularly simple proof of the "discrete separation theorem" of FRANK [4].

At this point we note that a function ℓ is *supermodular* on X and Y if

$$\ell(X) + \ell(Y) \leq \ell(X \cup Y) + \ell(X \cap Y).$$

THEOREM 4.1 (Discrete separation theorem)

Let C,L be intersecting families of subsets of E and c and ℓ be functions which are submodular and supermodular on intersecting pairs of C and L respectively. There exists a modular function x: $2^E \rightarrow R$ such that

$x(Y) \leq c(Y)$ *for* Y *in* C *and* $x(Y) \geq \ell(Y)$ *for* Y *in* L *if and only if*

$$\sum \ell(X_k) \leq \sum c(Y_k)$$

holds for any disjoint members X_k *of* L *and* Y_k *of* C *such that* $\dot{U}X_k = UY_k$. *More-over, if* c *and* ℓ *are integer-valued then* x *can be chosen to be integer-valued.*

PROOF. Let us assume that either $\emptyset \notin C$ and L or else that $\ell(\emptyset) = c(\emptyset) = 0$. (If this is not so, replace ℓ and c by $\ell' = \ell - \ell(\emptyset)$, $c' = c - c(\emptyset)$ and carry out the following argument to find a modular x' for ℓ' and c'.) Let M be a sufficiently large integer such that

$$c_1^+(X) = M|X| + c(X) \geq 0 \quad \text{for all } X \in C$$

$$c_2^+(X) = M|X| - \ell(X) \geq 0 \quad \text{for all } X \in L.$$

The functions c_1 and c_2 are nonnegative and submodular on all intersecting pairs in C and L respectively. Now construct a flow network as in Figure 1. The arcs directed into nodes 1 and 2 have capacity constraints defined by (C, c_1) and (L, c_2). Each arc from the source s has capacity 2M. The flows through the two arcs to the sink t are not subject to any capacity constraints.

Suppose there exists a flow x saturating each of the arcs from the source, i.e. a flow of value $2M|E|$. Let $x_1(e)$, $x_2(e)$ be the flows through the pair of arcs directed from the center node identified with the element $e \in E$. Then

$$x_1(e) + x_2(e) = 2M$$

and

$$x_1(Y) - M|Y| = M|Y| - x_2(Y),$$

for all $Y \subseteq E$. But $x_1(Y) \leq c_1(Y) = M|Y| + c(Y)$ for all Y in C and $x_2(Y) \leq c_2(Y) = M(Y) - \ell(Y)$ for all Y in L. It follows that

$$x(e) = x_1(e) - M = M - x_2(e), \quad \text{for } e \in E$$

yields the desired modular function and x can be assumed to be integral if ℓ and c are.

Now suppose that a maximal flow does not saturate all arcs directed from the source. Then there exists a dual structure (S,T,L,U) with c(S,T,L,U) < 2M|E|. In order for (S,T,L,U) to have finite capacity, nodes 1 and 2 must be in T and each arc (e,1),(e,2), where e $\in$ S, must be in L. Suppose a minimum capacity cut has center nodes S' $\subseteq$ E in S. Then the capacity of the cut is

$$2M|E-S'| + d_1^+(S') + d_2^+(S') < 2M|E|.$$

It follows that we have found a set $S' = UX_k = UY_k$ such that $\sum \ell(X_k) > > \sum c(Y_k)$. $\square$

It follows from the comments concerning crossing families in Section 2 that a more general version of the theorem can be obtained for crossing families. (One must consider all k such that x(E) = k.)
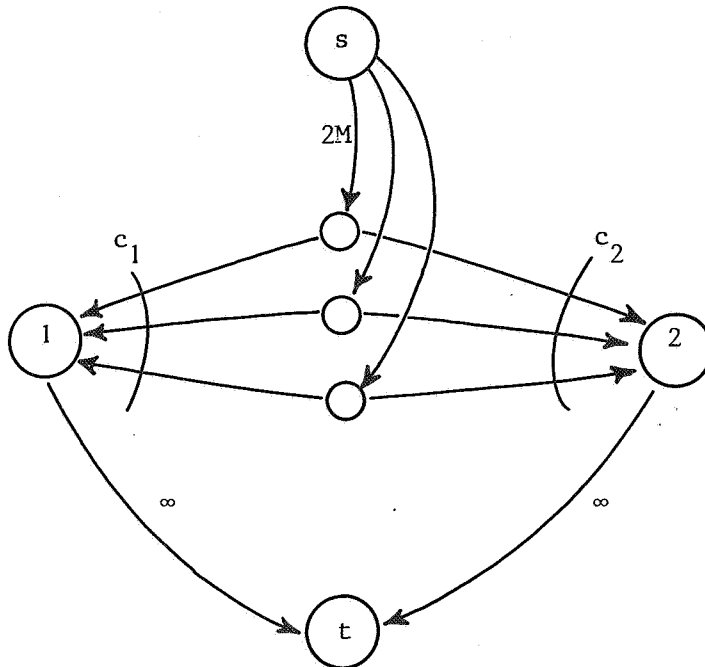

Figure 1. Network for proof of discrete separation theorem.

5. FLOWS WITH LOWER BOUNDS

As in the case of ordinary network flow theory, it is natural to consider nonzero lower bounds on arc flows. In order for lower bound constraints to have the same desirable properties with respect to tight sets as capacity constraints, it is natural to require the lower bound constraints to be supermodular rather than submodular. Hence, in addition to capacity constraints specified by intersecting families C_i^+, C_i^- and capacity functions c_i^+, c_i^- submodular over intersecting pairs, we also specify two intersecting families L_i^+, L_i^- for each node with functions ℓ_i^+, ℓ_i^- supermodular over intersecting pairs.

Now in order for a flow x to be feasible, we no longer require that x be nonnegative, but rather that

$$x(Y) \leq c_i^+(Y) \qquad \text{for all i and } Y \in C_i^+$$

$$x(Y) \geq \ell_i^+(Y) \qquad \text{for all i and } Y \in L_i^+,$$

and similarly for C_i^-, L_i^-. Note that the values $c_i^+(Y)$, $\ell_i^-(Y)$ may be arbitrary in sign. Hence in order for a flow to be feasible with respect to both (C_i^+, c_i^+), (L_i^+, ℓ_i^+) it must in effect be in the intersection of a certain extended polymatroid and a certain "contrapolymatroid."

Unfortunately, the capacity constraints and the lower bound constraints must be in a special relationship to each other, else the integrality theorem will fail to remain valid and it will become an NP-hard problem even to find a feasible integral flow, if one exists. The following simple example illustrates this point.

Consider the problem of determining whether or not there is a Hamilton path from node a to node b in a given directed graph G = (N,A) where without loss of generality we assume that the indegree of a and the outdegree of b are zero. A set ot arcs P constitutes such a path if and only if

(i)   P is an independent set of the graphic matroid induced by G (ignoring directions of arcs),

(ii) P contains no more than one arc into any given node, and

(iii) P contains at least one arc out of each node, except b.

We now form a flow network with only a source s, a sink t, and an arc from s to t for each arc of G. Conditions (i) and (ii) are enforced by letting $C_s^- = 2^A$, and c_i^- be the rank function of the graphic matroid. Conditions (ii) are enforced by letting $C_t^+ = \{A_i^+ \mid i \in N, i \neq a\}$ and $c_t^+(Y) = 1$ for all Y in C_t^+. For conditions (iii), we let $L_t^+ = \{A_i^- \mid i \in N, i \neq b\}$ and $\ell_t^+(Y) = 1$ for all Y in L_t^+. It follows that there exists a feasible integral flow in this network if and only if G has a Hamilton path from a to b.

The special relationship which we shall require for each pair (C_i^+, c_i^+), (L_i^+, ℓ_i^+) and (C_i^-, c_i^-), (L_i^-, ℓ_i^-) is as follows (dropping sub and superscripts): If $X \in C$, $Y \in L$ and $X \cap Y \neq \emptyset$ then $X - Y \in C$, $Y - X \in L$ and

(5.1) $c(X) - \ell(Y) \geq c(X-Y) - \ell(Y-X)$.

(A similar relation appears in the work of HASSIN [7,8] and also FRANK [6].) If this relation is satisfied we say that the capacity constraints (C,c) and lower bound constraints (L,ℓ) are *compliant*.

Let x be a flow which is feasible with respect to constraints (C,c) and (L,ℓ). Let us say that a set X in C is *c-tight* if $x(X) = c(X)$ and a set Y in L is *ℓ-tight* if $x(Y) = \ell(Y)$. If e is contained in any c-tight (ℓ-tight) set, let $C(e)(L(e))$ denote the unique minimal tight set in which e is contained.

LEMMA 5.1 *Let x be flow which is feasible with respect to a compliant pair of constraints $(C,c)(L,\ell)$. If e is contained in a c-tight set, then e is contained in any ℓ-tight set intersecting C(e). If e is contained in an ℓ-tight set, then e is contained in any c-tight set intersecting L(e).*

The lemma suggests a natural redefinition of augmenting paths:

(5.2) if the head(tail) of a backward arc in the path is contained in an ℓ-tight set then the preceding (following) arc is a forward arc in its unique minimal ℓ-tight set.

(5.3) if the head(tail) of a forward arc in the path is contained in a c-tight set, then the following (preceding) arc is a backward arc in its

unique minimal c-tight set.

It is apparent that, given a feasible flow, augmenting (or "deaugmenting") paths can be constructed by a labeling procedure and that a maximum value (or minimum value) flow can be computed. As a by-product of this procedure, we can obtain a max-flow min-cut (or min-flow max-cut) theorem. The dual structure (S,T,L,U) is the same as before, namely a bipartition S,T of the nodes plus a bipartition L,U of the arcs of the network. However we must modify the definition of capacity:

$$c(S,T,L,U) = \sum_{i \in S} d_i^- (A_i^- \cap \ ) - \sum_{j \in S} m_j^+ (A_j^+ \cap U)$$

$$+ \sum_{j \in T} d_j^+ (A_j^+ \cap L) - \sum_{i \in T} m_i^- (A_i^- \cap L)\cdot,$$

where d_i^-, d_j^+ all computed as before and

$$m_j^+(X) = \max \{ \sum \ell_j^+(X_k)\} \ ,$$

$$m_i^-(X) = \max \{ \sum \ell_i^-(X_k)\} \ .$$

THEOREM 5.2 (Max-flow min-cut theorem)
If all pairs of capacity constraints and lower bound constraints are compliant, then the maximum value of a flow is equal to the minimum capacity of a dual structure. Moreover,
(i) *if all constraints are specified by ring families, then* $d_j^+ = c_j^+$,
   $m_i^- = \ell_i^-$, $d_i^- = c_i^-$, $m_j^+ = \ell_j^+$.
(ii) *if all capacity constraints are specified by extended polymatroids and all lower bound constraints by contrapolymatroids, then L and U may be taken to be simply a bipartition of the (forward and backward) arcs across the cut* (S,T).

6. FINDING A FEASIBLE FLOW

We now turn to the problem of finding a feasible flow in a network with compliant capacity and lower bound constraints. This can be done

by a technique which is similar to one which is well known for ordinary flow networks.

As a first step, find for each node i two flows x_i^+ and x_i^- which are feasible with respect $(C_i^+, c_i^+), (L_i^+, \ell_i^+)$ and $(C_i^-, c_i^-), (L_i^-, \ell_i^-)$ respectively. This can be accomplished with 2n max-flow computations as described in Section 3. (If no such flows exist, then clearly there is no feasible flow for the network.)

If we are so fortunate as to have found flows such that

$$(6.1) \qquad x_i^+(A_i^+) = x_i^-(A_i^-) \quad \text{for } i \neq s,t,$$

$$(6.2) \qquad x_i^-(e) = x_j^+(e) \quad \text{for each arc } e = (i,j),$$

then we are done. In general, however, at least one of the equations (6.1), (6.2) will be violated.

We now subdivide each arc of the network and form a new flow network with dummy source s' and dummy sink t', as shown in Figure 2.

For each node i of the original network (including s,t), if $x_i^+(A_i^+) > x_i^-(A_i^-)$ we provide an arc of capacity $x_i^+(A_i^+) - x_i^-(A_i^-)$ from s' to i and no lower bound. Similarly, if $x_i^+(A_i^+) < x_i^-(A_i^-)$ we provide an arc of capacity $x_i^-(A_i^-) - x_i^+(A_i^+)$ from i to t'. For each arc e = (i,j) such that $x_i^-(e) > x_j^+(e)$ we provide an arc of capacity $x_i^-(e) - x_j^+(e)$ from s' to the node created by subdivision of e. Similarly we provide an arc to t' if $x_i^-(e) < x_j^+(e)$. We also provide an arc (t,s) with no capacity or lower bound constraint.

Each of the original capacity functions $c_i^+(c_i^-)$ is replaced by a capacity function $c_i^+ - x_i^+ (c_i^- - x_i^-)$, and each lower bound function $\ell_i^+ (\ell_i^-)$ is replaced by $\ell_i^+ - x_i^+ (\ell_i^- - x_i^-)$. Note that each new capacity function is nonnegative and each new lower bound function is nonpositive. It follows that the zero flow is feasible, and we can proceed to compute a maximal flow from s' to t' using the redefined concept of an augmenting path given in the previous section.
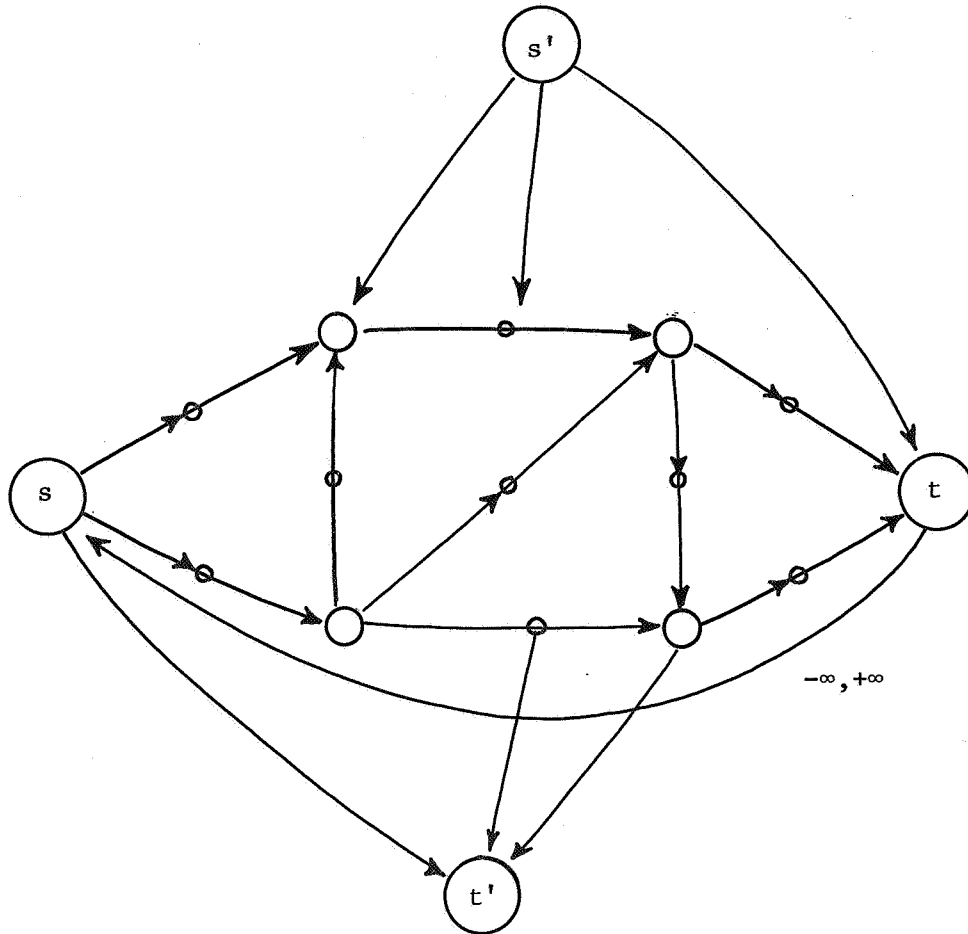
Figure 2. Network for computation of feasible flow.

Suppose there exists a flow x' in the new network which saturates each of the arcs directed from s' (and hence also each of the arcs directed into t'). We assert that now $x + x'$ is a feasible flow in the original network.

If there does not exist a saturating flow in the new network, then one can apply the max-flow min-cut theorem of the previous section to construct a subset S' of nodes such that the lower bounds require more flow into S' than the capacity constraints allow to flow out.

7. FORMULATION OF THE EDMONDS-GILES PROBLEM

The Edmonds-Giles problem is as follows. Let $G = (N,A)$ be a digraph,

(C,c) be a crossing family defined over subsets of N, with C submodular on crossing pairs. Suppose there are given a "capacity" function c': $A \longrightarrow R$, a "lower bound" function ℓ: $A \longrightarrow R$ and a "weight" function w: $A \longrightarrow R$. Then the problem is to find a "flow" x to maximize wx subject to

$$\ell \leq x \leq c'$$

and constraints such that for each Y in C the loss of flow is at most $c(Y)$, where the *loss* means the total amount of flow leaving Y minus the total amount of flow entering Y. Note that it is *not* required that at each node of G the amount of flow entering that node is equal to the amount of flow leaving the node.

This problem can be converted to a circulation problem by a simple transformation. Create a node s with an arc from s to each node of G. Then any circulation in the network corresponds to a "flow" in G. Identifying arcs from s with nodes, let $C_s^- = C$ and $c_s^-(Y) = c(Y)$ for each in Y in C. Then any circulation feasible with respect to (C_s^-, c_s^-) and the individual arc capacities and lower bounds c', ℓ corresponds to a feasible solution to the Edmonds-Giles problem.

Since $x(A_s^-) = 0$, we can apply the transformation described in Section 2 to replace the crossing family of constraints (C_s^-, c_s^-) by an intersecting family. A feasible solution to the Edmonds-Giles problem can be computed by a single maximal flow computation, using the technique described in the previous section. (Note particularly that to find a feasible flow in the network shown in Figure 2 we can set $x(e) = - M$, where M is a sufficiently large integer, for each $e \in A_0^-)$. The author believes that this approach is both cleaner and more general than that proposed by FRANK [4] . The only "practical" computational problems are those of determining minimal c_s^- - tight subsets and finding the amount of augmentation to be made along each augmenting path. How these problems are dealt with depends on the exact nature of the crossing family of constraints (C,c).

It should be noted that the principal theorem of Edmonds and Giles is that their problem has an optimal primal solution in integers if ℓ, c'

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and c are integer and that it has an optimal dual solution in integers if
w is integral. A similar theorem can be obtained for more general maximum
weight (or minimum cost) circulation problems involving compliant capacity
and lower bound constraints. (Such a theorem is virtually explicit in the
work of HASSIN [7,8] .) The Edmonds-Giles theorem then becomes a corollary
of the circulation theorem. (Note that since the transformation from crossing
constraints to intersecting constraints is integer-preserving, integrality
theorems for intersecting constraints imply similar integrality theorems for
crossing constraints.)

8. A FURTHER GENERALIZATION

A fairly obvious generalization of the network flow mode is to specify
only a single set of constraints (C_i, c_i) on $A_i^+ \cup A_i^-$. The interpretation of
such constraints is as follows. If Y is in C_i then the *net flow* $\bar{x}(Y)$ into
node i through Y must not exceed c(Y), i.e.

$$\bar{x}(Y) = x(Y \cap A_i^+) - x(Y \cap A_i^-) \le c(Y).$$

Similarly, if (L_i, ℓ_i) specifies lower bound constraints,

$$\bar{x}(Y) = x(Y \cap A_i^+) - x(Y \cap A_i^-) \ge \ell(Y)$$

for all Y in L.

If i is a node for which the total flow in must equal the total flow
out, then there is no need for both capacity constraints and lower bound
constraints, since $\bar{x}(Y) \ge \ell(Y)$ if and only if $\bar{x}(\bar{Y}) \le - \ell(Y)$. We can then set

$$C_i' = C_i \cup \{ Y | \bar{Y} = (A_i^+ \cup A_i^-) - Y \in L_i \}$$

and $c_i'(Y) = c_i(Y)$ for Y in C_i and $c_i'(Y) = - \ell_i(Y)$ for $\bar{Y}$ in L_i. If (C_i, c_i),
(L_i, ℓ_i) are intersecting constraints which are compliant then C_i' is an in-
tersecting family and c_i' is submodular on intersecting pairs.

(Apply the relations $X \cap Y = X - \bar{Y}$, $X \cup Y = \overline{(\bar{Y}-X)}$.)

Note that this transformation can be applied to convert compliant intersecting constraints $(C_i^+, c_i^+), (L_i^+, \ell_i^+)$ and $(C_i^-, c_i^-), (L_i^-, \ell_i^-)$ to a single intersecting family of constraints (C_i', c_i'). (Here (C_i^+, c_i^+) and (L_i^-, ℓ_i^-) define (C_i, c_i) and $(L_i^+, \ell_i^+), (C_i^-, c_i^-)$ define (L_i, ℓ_i).)

If node i is not such that the total flow in must equal the total flow out, then it is not possible to apply the above transformation and it remains useful to have two intersecting families of constraints (C_i, c_i) and (L_i, ℓ_i). If all such pairs of constraints are compliant, the theory remains very much as before. The definition of an augmenting path remains very much the same as in Section 5:

(8.1) if the head (tail) of a backward arc in the path is contained in an ℓ-tight (c-tight) set then the preceding (following) arc is contained in its unique minimal ℓ-tight (c-tight) set, and

(8.2) if the head (tail) of a forward arc in the path is contained in a c-tight (ℓ-tight) set, then the following arc is an arc in its unique minimal c-tight (ℓ-tight) set.

The max-flow min-cut theorem also goes through very much as before.

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